Exam 2 is next Thursday!

Covers: 10.1-10.3: Analyzing a function
11.1,2: Deriv. of ln(x) and e^x
12.1,3,4: Integrals, finding C
13.2: Definite Integrals

Be able to answer any question about:
critical points, increasing/decreasing, local max/min, global max/min, concave up/down, inflection points, horizontal tangents

12.4: Antiderivatives and Applications

First, let's discuss how to find "C".

Entry Task: (from 12.4 HW)

Suppose $MR(q) = 14 - 0.2q$.

$$MC(q) = 30\sqrt{q+4}.$$

and fixed costs are $FC = \$900$.

Find the formulas for $TR(q)$ and $TC(q)$.

$TR(q) = \int MR(q)\,dq$

$= \int 14 - 0.2q\,dq$

$= 14q - 0.2\frac{1}{2}q^2 + C$

$\boxed{TR(q) = 14q - 0.1q^2 + C}$

Since $TR(0) = 0$ ← ALWAYS TRUE!

$14(0) - 0.1(0)^2 + C = 0 \Rightarrow C = 0$

$\boxed{TR(q) = 14q - 0.1q^2}$ CHECK!!!

$TC(q) = \int 30(q+4)^{1/2}\,dq$

$TC(q) = 30\frac{1}{3/2}(q+4)^{3/2} + C$

$TC(q) = 20(q+4)^{3/2} + C$

$TC(0) = 900$ ← GIVEN

So $20(0+4)^{3/2} + C = 900$

$\Rightarrow 20 \cdot 8 + C = 900$

$\Rightarrow 160 + C = 900 \Rightarrow C = 740$

$\boxed{TC(q) = 20(q+4)^{3/2} + 740}$ CHECK!!!

Note: To integral $MC(q)$ in the entry task, you had to guess and check a slightly varied version of the formula:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C,$$

which is this

$$\int (x+a)^n \, dx = \frac{1}{n+1} (x+a)^{n+1} + C$$

You won't use this a lot, but are welcome to make a note of this more general version. This is the only slight variation from our four examples that you will see in homework.

Please remember to always, always, check your antiderivatives (by differentiating)!!

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

$$\int k \, dx = kx + C$$

$$\int \frac{1}{x} \, dx = \ln(x) + C$$

HOW TO FIND "C"

Step 1: Integrate (don't forget +C)

Step 2: Plug in your initial condition.

(The given x and y values)

Step 3: Solve for C.

Example:

$$f'(x) = 8e^{4x} + \frac{1}{\sqrt{x}}\ , \quad \text{and} \quad f(1) = 15$$

Find $f(x)$.

$$f(x) = \int 8e^{4x} + x^{-1/2}\,dx$$

$$= 8 \cdot \frac{1}{4} e^{4x} + \frac{1}{1/2} x^{1/2} + C$$

$$= 2e^{4x} + 2\sqrt{x} + C$$

$$f(1) = 15 \Rightarrow 2e^4 + 2 + C = 15$$

$$\Rightarrow C = 13 - 2e^4$$

$$\boxed{f(x) = 2e^{4x} + 2\sqrt{x} + 13 - 2e^4}$$

CHECK!!!

Example:

$$f''(x) = -32\ , f'(0) = 0, f(0) = 100$$

Find $f(x)$.

$$f'(x) = \int -32\,dx$$

$$f'(x) = -32x + C$$

$$f'(0) = 0 \Rightarrow -32(0) + C = 0$$

$$\Rightarrow C = 0$$

So $f'(x) = -32x$

$$f(x) = \int -32x\,dx$$

$$= -16x^2 + C$$

$$f(0) = 100 \Rightarrow -16(0)^2 + C = 100$$

$$C = 100$$

$$\boxed{f(x) = -16x^2 + 100}$$

How to do all applications problems

Step 1: What are you given?

Identify, label and use the connections to find all related functions.

Step 2: What do you want?

What function is the question asking about? Write that function down first!!! That is your “original” function for this problem.

Step 3: Translate. Solve.
I gave two extensive handouts on how to translate and solve and you’ve done dozens in homework. You should know the methods well.

Step 4: Present your answer.
Is the question asking for the *value* of the function (meaning output), or the input, *x*, where it occurs, or something else? Read carefully and appropriately interpret your work.

critical number, horizontal tangent
$f'(x) \stackrel{?}{=} 0$ solve

increasing / decreasing
$f'(x) = 0$ solve
number line
$\nearrow \quad \searrow \quad \nearrow$
$+ \quad - \quad +$

local max/min
$f'(x) = 0$ solve
number line <u>or</u> 2nd deriv. test
$f'(a) = 0$ & $f''(a) > 0$ $\cup$
$f'(a) = 0$ & $f''(a) < 0$ $\cap$

concave up/down, pts of inflection
$f''(x) = 0$ solve
number line
$\cap \quad \cup \quad \cap$
$- \quad + \quad -$

global max/min
$f'(x) = 0$ solve
plug critical pts & endpoints into original function

Example: (Old Exam Question – like HW)
Two vats have water coming in and out.
At time t hours, we define:

$A(t)$ = "gallons of water in Vat A"
$B(t)$ = "gallons of water in Vat B"

You are given

$A'(t) = -3t^2 + 24t - 21$ gal/hr
$B(t) = 3t - 9\ln(t+1) + 10$ gallons

The two vats contain the same amount of water at time $t = 0$.

(a) Find the formula for A(t).

(b) Find and classify all critical numbers of A(t).

(c) Find the global maximum of B(t) on the interval t = 0 to t = 5.

(d) Give the longest interval over which the graph of A(t) is concave up.

(a) $A(t) = \int -3t^2 + 24t - 21\,dt$

$= -3\frac{1}{3}t^3 + 24\frac{1}{2}t^2 - 21t + C$

$A(t) = -t^3 + 12t - 21t + C$

AND $A(0) = \underbrace{B(0)}_{3(0) - 9\ln(1) + 10 =} = 10$

$\Rightarrow -(0)^3 + 12(0) - 21(0) + C = 10 \Rightarrow C = 10$

$\boxed{A(t) = -t^3 + 12t - 21t + 10}$ CHECK!!!

(b) WANT CRITICAL # of A(t). ← original function

FIND $A'(t) = -3t^2 + 24t - 21 \stackrel{?}{=} 0$

$-3(t^2 - 8t + 7) \stackrel{?}{=} 0$

$-3(t-1)(t-7) \stackrel{?}{=} 0$

$t = 1$ or $t = 7$

2 ways to classify

- 1st Deriv. Test

$\searrow \quad \nearrow \quad \searrow$

A′: $-$ | 1 | $+$ | 7 | $-$

- 2nd Deriv. Test

$A''(t) = -6t + 24$

$A''(1) = 18 > 0$ $\cup$ ← min

$A''(7) = -18 < 0$ $\cap$ ← max

(c) WANT GLOBAL MAX OF $B(t)$ ON $t=0$ to $t=5$

"original"

FIND $B'(t) = 3 - 9 \cdot \frac{1}{t+1} \cdot 1 \stackrel{?}{=} 0$

$$3(t+1) - 9 = 0$$
$$3t + 3 - 9 = 0$$
$$3t = 6$$
$$t = 2$$

$B(0) = 10$

$B(2) = 3(2) - 9\ln(2+1) + 10$
$= 16 - 9\ln(3) \approx 6.1125$

$B(5) = 3(5) - 9\ln(5+1) + 10$
$= 25 - 9\ln(6) \approx 8.8742$

$\boxed{\text{GLOBAL MAX} = 10}$

$\boxed{\text{GLOBAL MIN} = 16 - 9\ln(3)}$

(d) WANT INTERVAL WHERE $A(t)$ IS CONCAVE UP.

original

$A''(t) = -6t + 24 \stackrel{?}{=} 0$

$$-6t = -24$$
$$t = 4$$

A' +   4   −

$\boxed{t < 4}$

Business Application Review:

We have ways to go between *all* our business functions!

HW 12.4 Overview:

1. MR to TR.
2. AC' to AC.
3. MR to TR.
4. MR/MC to TR/TC.

 etc….

Total Values and Marginal Values	Total Values and Average Values
$TR(x) = \int MR(x)dx$ $TR'(x) = MR(x)$	$AR(x) = \dfrac{TR(x)}{x} = price$ $TR(x) = xAR(x)$
$TC(x) = \int MC(x)dx$ $TC'(x) = MC(x)$	$AC(x) = \dfrac{TC(x)}{x}$ $TC(x) = xAC(x)$
$P(x) = \int MP(x)dx$ $P'(x) = MP(x)$	

Initial conditions: $TR(0) = 0,$ $TC(0) = FC$

$P(x) = TR(x) - TC(x)$

Example (from HW):

$$AC'(q) = -\frac{4}{q^2} + \frac{1}{4},$$

$$AC(4) = 10$$

Find the formula for $AC(q)$ and $TC(q)$. what is FC?

$$AC(q) = \int AC'(q)\,dq = \int -4q^{-2} + \tfrac{1}{4}\,dq = -4\tfrac{1}{-1}q^{-1} + \tfrac{1}{4}q + C$$

$$AC(q) = \frac{4}{q} + \frac{1}{4}q + C$$

$$AC(4) = 10 \Rightarrow \frac{4}{(4)} + \frac{1}{4}(4) + C = 10 \Rightarrow 2 + C = 10 \Rightarrow \boxed{C = 8}$$

$$\boxed{AC(q) = \frac{4}{q} + \frac{1}{4}q + 8}$$

$$\boxed{TC(q) = q\,AC(q) = q\left(\frac{4}{q} + \frac{1}{4}q + 8\right) = 4 + \frac{1}{4}q^2 + 8q}$$

NOTE: $TC(0) = 4 = FC$

1. (16 points) Let $f(x) = 4x^3 - 78x^2 + 432x$.

   (a) Find all critical numbers of $f(x)$ and use the Second Derivative Test to determine whether each gives a local maximum or a local minimum value of $f(x)$.

$\boxed{\text{STEP 1}}$ $f'(x) = 12x^2 - 156x + 432 \overset{?}{=} 0$ FACTOR OUT 12

$12(x^2 - 13x + 36) \overset{?}{=} 0$

$12(x-4)(x-9) \overset{?}{=} 0 \Rightarrow \boxed{x = 4 \text{ or } x = 9}$ OR USE QUAD. FORMULA

$\boxed{\text{STEP 2}}$ $f''(x) = 24x - 156$

At $x = 4$, $f''(4) = 24(4) - 156 = -60$: $\left.\begin{matrix} f'(4) = 0 \\ f''(4) < 0 \end{matrix}\right\}$ $\boxed{\text{LOCAL MAX}}$

At $x = 9$, $f''(9) = 24(9) - 156 = 60$: $\left.\begin{matrix} f'(9) = 0 \\ f''(9) > 0 \end{matrix}\right\}$ $\boxed{\text{LOCAL MIN}}$

ANSWER: $x =$ 4 gives a local MAX

ANSWER: $x =$ 9 gives a local MIN

   (b) Define a new function $D(x)$ by $D(x) = \frac{f(x)}{x}$. Find the value of x at which $D(x)$ reaches its smallest value. (Your work should include an explanation of how you know $D(x)$ is smallest there.)

$\boxed{\text{STEP 1}}$ $D(x) = 4x^2 - 78x + 432 \Rightarrow D'(x) = 8x - 78 \overset{?}{=} 0$

$8x = 78 \Rightarrow x = \frac{78}{8} = 9.75$

$\boxed{\text{STEP 2}}$ D: $\searrow$ $\nearrow$

D': $-$ 9.75 $+$

For $x < 9.75$, $D'(0) = -78 < 0$

For $x > 9.75$, $D'(10) = 8(10) - 78 = 2 > 0$

$\Big\}$ THUS, AT $x = 9.75$ $D(x)$ MUST REACH ITS SMALLEST VALUE

ANSWER: $x =$ 9.75

   (c) Define a new function $S(x)$ by $S(x) = \frac{D(x)}{x}$. Find all positive critical numbers of $S(x)$.

$\boxed{\text{STEP 1}}$ $S(x) = 4x - 78 + \frac{432}{x} = 4x - 78 + 432x^{-1}$

$\Rightarrow S'(x) = 4 - 432x^{-2} = 4 - \frac{432}{x^2} \overset{?}{=} 0$

$\Rightarrow 4x^2 - 432 = 0 \Rightarrow 4x^2 = 432 \Rightarrow x^2 = 108$

$x = \pm\sqrt{108}$

ANSWER: $x =$ $\sqrt{108} \approx 10.392$

2. (19 points) You sell Gizmos. Your total revenue and total cost are given by the functions $TR(q) = -2q^2 + 199.1q$ and $TC(q) = 0.01q^3 - 2.405q^2 + 200q + 20$, where q is in thousands of Gizmos and TR and TC are both in thousands of dollars.

(a) Find the largest interval on which $MR(q)$ is positive.

$\boxed{\text{STEP 1}}$ $MR(q) = -4q + 199.1 \overset{?}{=} 0 \Rightarrow 199.1 = 4q \Rightarrow q = \frac{199.1}{4} = 49.775$

$\boxed{\text{STEP 2}}$ Tr $\nearrow$ $\searrow$

MR $(+)$ 49.775 $-$

ANSWER: from $q =$ 0 to $q =$ 49.775 thousand Gizmos

(b) Is $TC(q)$ concave up or concave down at $q = 100$?

$\boxed{\text{STEP 1}}$ $TC'(q) = 0.03q^2 - 4.81q + 200$

$TC''(q) = 0.06q - 4.81$

$\boxed{\text{STEP 2}}$ $TC''(100) = 0.06(100) - 4.81 = 6 - 4.81 = 1.19 > 0$

ANSWER: (circle one) concave up concave down

(c) Recall that $FC = TC(0)$, $TC(q) = VC(q) + FC$, and $AVC(q) = \dfrac{VC(q)}{q}$. Find all critical numbers of $AVC(q)$.

$\boxed{\text{STEP 1}}$ $AVC(q) = \dfrac{0.01q^3 - 2.405q^2 + 200q}{q} = 0.01q^2 - 2.405q + 200$

$\Rightarrow AVC'(q) = 0.02q - 2.405 \overset{?}{=} 0$

$q = \dfrac{2.405}{0.02} \approx 120.25$

ANSWER: (list all) $q =$ 120.25 thousand Gizmos

(d) Let $P(q)$ denote the profit (in thousands of dollars) at q thousand Gizmos. The critical numbers of $P(q)$ are $q = 1.16$ and $q = 25.84$ thousand Gizmos. Determine whether each critical number gives a local minimum of $P(q)$, a local maximum of $P(q)$, or neither.

$\boxed{\text{Step 1}}$ $P'(q) = (-4q + 199.1) - (0.03q^2 - 4.81q + 200)$

$\Rightarrow P'(q) = -0.03q^2 + 0.81q - 0.9$

$P''(q) = -0.06q + 0.81$

$\boxed{\text{Step 2}}$ $q = 1.16 \Rightarrow P''(1.16) = -0.06(1.16) + 0.81 = 0.7404 > 0$ $\cup \leftarrow$ Local min

$q = 25.84 \Rightarrow P''(25.84) = -0.06(25.84) + 0.81 = -0.7404 < 0$ $\cap$ Local max

ANSWER: $q = 1.16$ gives a (circle one) local min local max neither

$q = 25.84$ gives a (circle one) local min local max neither

4. (13 pts) Your Total Cost (in hundreds of dollars) and Demand Curve (in dollars) *vs.* the quantity q in hundreds of Items sold is given by the function:

$$TC(q) = \frac{q^3}{12} - \frac{q^2}{2} + \frac{3}{4}q + 10 \quad \text{and} \quad p = h(q) = 24 - 8\sqrt{q}.$$

(a) (7 pts) Write the formula for **Total Revenue**, TR, and give the **prices** that correspond to the global maximum and global minimum value of **Total Revenue** over the interval $q = 2$ to $q = 6$ hundred Items.

$\boxed{\text{STEP 1}}$ $TR(q) = 24q - 8q^{3/2}$

$TR'(q) = 24 - 8 \cdot \frac{3}{2} q^{1/2} = 24 - 12q^{1/2} \overset{?}{=} 0$

$q^{1/2} = 2 \Rightarrow q = 4$

$\boxed{\text{STEP 2}}$ $TR(2) = 24(2) - 8(2)^{3/2} \approx 25.3726 \leftarrow$ MIN

$TR(4) = 24(4) - 8(4)^{3/2} = 32 \leftarrow$ MAX

$TR(6) = 24(6) - 8(6)^{3/2} = 26.4245$

$\boxed{\text{STEP 3}}$ CORRESPONDING PRICE?

$p = h(2) = 24 - 8\sqrt{2} = 12.687$

$p = h(4) = 24 - 8\sqrt{4} = 8$

ANSWER: **PRICE** for the global minimum value = 12.69 dollars

PRICE for the global maximum value = 8 dollars

(b) (6 pts) Find **all** critical numbers of **Total Cost**, TC. Then use the second derivative test to determine whether $TC(q)$ reaches a local maximum, local minimum, or tell me if the test is inconclusive. Clearly put a box around your critical numbers and clearly label each as either local max, local min, or test inconclusive.

$\boxed{\text{STEP 1}}$ $TC'(q) = \frac{1}{4}q^2 - q + \frac{3}{4} \overset{?}{=} 0 \overset{\cdot 4}{\Rightarrow} q^2 - 4q + 3 = 0$

$(q-1)(q-3) = 0$

OR USE QUAD. FORMULA

$\boxed{\text{STEP 2}}$ $TC''(q) = \frac{1}{2}q - 1$

At $q = 1$, $TC''(1) = \frac{1}{2}(1) - 1 = -\frac{1}{2} < 0$ $\frown$ LOCAL MAX

At $q = 3$, $TC''(3) = \frac{1}{2}(3) - 1 = \frac{1}{2} > 0$ $\smile$ LOCAL MIN

$$\boxed{\begin{array}{ll} q = 1 & \text{LOCAL MAX} \\ q = 3 & \text{LOCAL MIN} \end{array}}$$